

Spherical Orbifolds for Cosmic Topology

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Abstract. Harmonic analysis is a tool to infer cosmic topology from the measured astrophysical cosmic microwave background CMB radiation. For overall positive curvature, Platonic spherical manifolds are candidates for this analysis. We combine the specific point symmetry of the Platonic manifolds with their deck transformations. This analysis in topology leads from manifolds to orbifolds. We discuss the deck transformations of the orbifolds and give eigenmodes for the harmonic analysis as linear combinations of Wigner polynomials on the 3-sphere. These provide new tools for detecting cosmic topology from the CMB radiation.

1. Motivation and elements of topology.

Data observed by the Wilkinson satellite [15] and expected from the Planck satellite display the black-body cosmic microwave background CMB radiation with temperature $T = 2.75$ K and $\lambda_{\max} = 19$ mm, Fig. 1. This radiation is generally assumed to stem from the big bang. A small amplitude for low CMB multipoles l suggest selection rules. Do they arise from cosmic topology? In this first section we introduce topology and functional analysis for manifolds. Multiply connected spherical 3-manifolds can model the space part of cosmic space-time. We demonstrate

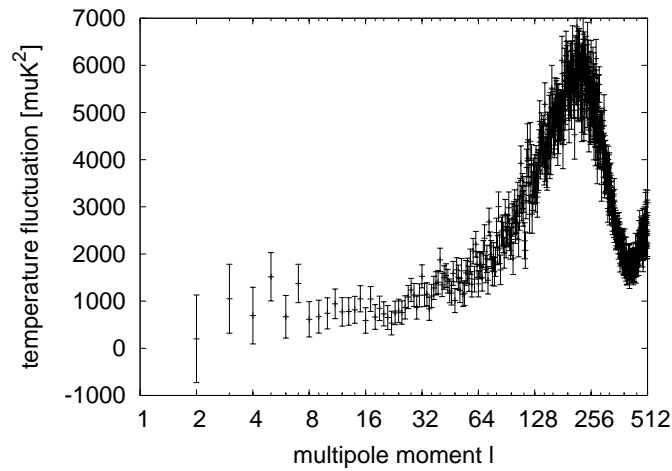


Figure 1. CMB Amplitude from the WMAP team [15] as a function of the multipole order l .

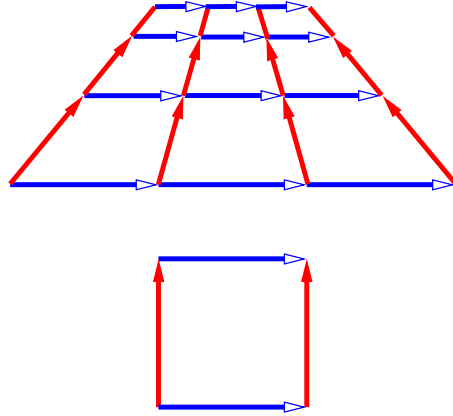


Figure 2. Parallel universes? Lift the square to its universal cover. By matching with horizontal/vertical neighbours we cover the plane by parallel squares.

topology in two dimensions from a square. First in Fig. 2 we lift this square to the Euclidean plane E^2 , called its universal cover. The square becomes part of a square tiling of E^2 . The group that generates the tiling is called the deck group. In a second approach, Fig. 3, we fold and glue the square into the torus. The gluings correspond to a group, termed the homotopy group H of the square. Homotopy and covering are linked by a theorem of Seifert and Threlfall [22] which says that the homotopy and the deck group are isomorphic. Fig. 3 illustrates the gluings, but may be misleading with respect to curvature: The square is Euclidean and shares zero curvature with its cover, the Euclidean plane. So the two views of a topological manifold, in terms of its universal cover or its self-gluing, allow to express different topological aspects and must be taken in conjunction.

1.1. Functions carried on a topological manifold.

We pass to functions carried by the square. On the covering plane, periodic boundary conditions on the square tiles yield twofold Fourier analysis. Any Fourier basis function

$$\psi_{n_1, n_2}(x, y) = \exp i(k_1 x + k_2 y), \quad k_1 L = 2\pi n_1, \quad k_2 L = 2\pi n_2, \quad (n_1, n_2) = 0, \pm 1, \pm 2, \dots \quad (1)$$

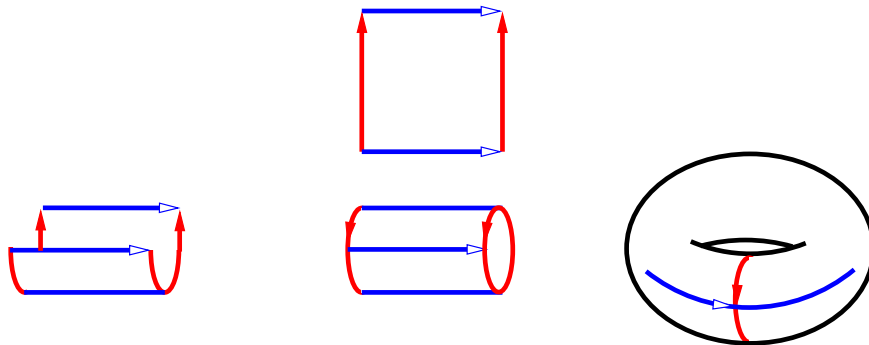


Figure 3. Folded universe? Glue the square by local homotopy. Horizontal edge folding plus gluing yields the cylinder. Successive vertical edge folding plus gluing gives the 2-torus T^2 .

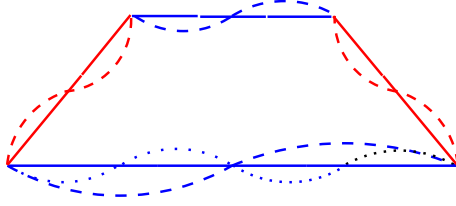


Figure 4. Periodic functions on the square from Fourier analysis.

is invariant under the deck group $H = C_\infty \times C_\infty$. As any deck group, H must act without fixpoints, compare Fig. 2.

A random function on a topological manifold is generated by a random superposition of its characteristic eigenmodes. In cosmic topology one compares such random superpositions with the observed CMB radiation. A similarity of both is thought to indicate the physical topology of the space part from space-time.

Before a comparison can be made, we must pass from the 2-dimensional square to 3-dimensional and more realistic manifolds. In contrast to zero curvature found for the square, positive curvature suggests to study spherical manifolds to which we turn in section 2.

The exploration of point symmetry for topology will lead us to the notion of orbifolds. In contrast to topological manifolds, orbifolds have deck group actions with singular fixpoints, that is, fixpoints of non-zero but finite order.

1.2. Einstein's field equations.

Einstein's field equations determine the metric structure of space-time from the energy-momentum tensor.

$$(R_{ik} - \frac{1}{2}g_{ik}) - \lambda g_{ik} = -\kappa T_{ik}, \quad (2)$$

$$\kappa = \frac{8\pi C}{c^4} = 23.07 \times 10^{-48} \frac{s^2}{g \text{ cm}}.$$

Here, R_{ik} is the Ricci tensor, g_{ik} the metrical tensor, T_{ik} the energy-momentum tensor, C Newton's gravitational, λ the cosmological constant. The Ricci tensor contains second derivatives of the metrical tensor, and so the field equations are second order partial differential equations for the metrical tensor g_{ik} in terms of a given energy-momentum tensor T_{ik} . Note: The cosmic mass distribution is smoothed out into a mass fluid!

Einstein in his first paper [4], as is also explained in [16], applied his equations to space taken as a 3-sphere. This choice assumes positive average curvature and closure, and still is relevant for present-day cosmic topology.

1.3. Selection rules from point symmetry and orbifolds.

Angular coordinates with respect to the midpoint of the square are

$$z = \rho \exp(i\phi), \quad \rho \geq 0, \quad 0 \leq \phi < 2\pi. \quad (3)$$

Functions in multipole form may be written as

$$f(z) = R_m(\rho) \exp(im\phi), \quad m = 0, \pm 1, \pm 2, \dots \quad (4)$$

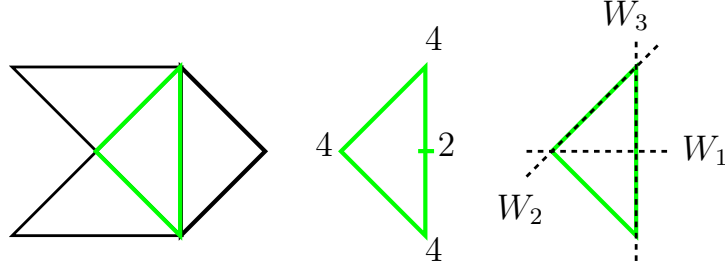


Figure 5. Orbifolds: The square is an orbit under the cyclic group C_4 of the triangular orbifold Δ . Left: Three Δ 's cover edges of Δ . Middle: Singular fixpoints of finite orders $(4, 4, 4, 2)$ on the edges $\partial\Delta$ of the orbifold Δ . Right: Three Weyl reflections W_1, W_2, W_3 in the dashed lines generate a Coxeter group Γ . Its rotational unimodular subgroup $ST : \langle W_1W_2, W_2W_3, W_3W_1 \rangle$ of even reflections generates a triangular tiling of the plane into orbifolds, preserves orientation, and yields selection rules. The unimodular subgroup $H = ST$ is the deck and homotopy group of the orbifold Δ !

Selection rules from point symmetry under the cyclic group C_4 then imply

$$m \equiv 0 \pmod{4} = 0, \pm 4, \pm 8, \dots \quad (5)$$

The introduction of point symmetry into topology leads to orbifolds rather than manifolds. For the square we explain the connection in Fig. 5. A triangle Δ forms a fundamental domain for the cyclic group C_4 acting on the square. The application of even products from a Coxeter group to a single preimage covers the plane E^2 by copies of the triangle. This action has rotational fixpoints on edges of the triangle, marked in Fig. 5, and so is not fixpoint-free as required for the deck transformation of a topological manifold. Instead, the triangle with its fixpoints forms an orbifold.

2. Spherical spaces.

After the introductory section 1 we turn now to realistic models of cosmic 3-space. For these models of cosmic space we must extend all the notions illustrated on the square to appropriate manifolds. The spaces should have dimension 3 and should have positive curvature like spherical spaces. Such spaces result by closing pieces from the 3-sphere as used by Einstein [4] for his initial cosmology. Our exposition now necessarily becomes more technical and uses results given in the references. Harmonic analysis on the manifolds aims at explaining the observed low amplitudes at small multipole order l of the cosmic microwave background. We analyze assumptions of point symmetry and randomness for Platonic spherical spaces. There emerge four new spaces, called orbifolds, with low volume fraction from the 3-sphere and sharp multipole selection rules in their eigenmodes.

3. Eigenmodes for spherical spaces.

Spherical spaces or spherical space forms are abstractly described as the quotients of the covering manifold S^3 by the deck group H of order $|H|$ acting fixpoint-free. The topologies are then classified as space forms $\mathcal{M} = S^3/H$. They were characterized with faithful representations of groups H by Wolf in [25] pp. 198-230. This characterization leaves open the geometry of the space, and so gives no access to point symmetry. Therefore we prefer a fully geometric approach [12], starting from homotopies derived by Everitt [5].

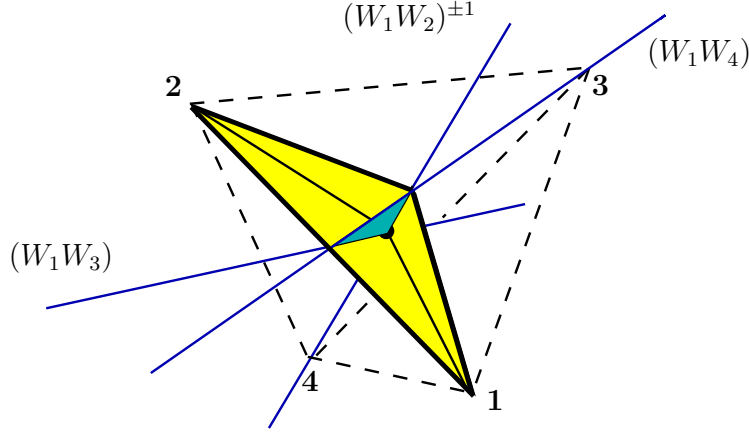


Figure 6. Tetrahedral orbifold. Face gluing deck rotations of the tetrahedral duplex orbifold (yellow) $N8$ in Euclidean version. The vertices of the Platonic tetrahedron are marked by the numbers (1, 2, 3, 4). A small tetrahedron of volume fraction $\frac{1}{12}$ inside the Platonic tetrahedron is the orbifold (yellow colour). The four covering rotations (W_1W_3) , $(W_1W_2)^{\pm 1}$, (W_1W_4) of this orbifold, corresponding to the covering of its four faces, are products of Weyl reflections from the Coxeter group with diagram $\Gamma = \circ - \circ - \circ - \circ$. They are marked by their rotation axes, intersection lines of pairs of Weyl reflection planes. The three intersections of these axes with the orbifold form the edges of the glue triangle (light blue area) of two Coxeter simplices. The inner points of its edges have the orders (2, 3, 2) of their covering rotations.

The 3-sphere S^3 of constant positive curvature $\kappa = 1$, Einstein 1917 [4], covers all spherical space forms. There is a one-to-one correspondence of S^3 and $SU(2, C)$ given by

$$\begin{aligned} x &= (x_0, x_1, x_2, x_3), \\ \leftrightarrow u(x) &= \begin{bmatrix} x_0 - ix_3 & -x_1 - ix_2 \\ x_1 - ix_2 & x_0 + ix_3 \end{bmatrix} \in SU(2, C), \\ x_0^2 + x_1^2 + x_2^2 + x_3^2 &= \det(u) = 1. \end{aligned} \quad (6)$$

The rotation group $SO(4, R)$ acts on the points of the 3-sphere by the correspondence

$$SO(4, R) = (SU^l(2, C) \times SU^r(2, C)) / \pm (e, e), \quad (7)$$

and its elements (g_l, g_r) transform these points by

$$(g_l, g_r) : u \rightarrow g_l^{-1} u g_r. \quad (8)$$

Wigner introduced the matrices $D_{m_1 m_2}^j(u)$ as the irreducible representation matrices of the group $SU(2, C)$ [3]. Considered as homogeneous polynomials in the four elements of the matrix $u(x)$, we call them Wigner polynomials. Their overall degree is $2j$ and is conserved under the linear transformations eq. 9. The Wigner polynomials are a complete and orthonormal set of functions on the 3-sphere S^3 . Moreover they are harmonic [12], that is, vanish under the Laplacian on the Euclidean space E^4 that embeds the 3-sphere.

The action of $SO(4, R)$ on Wigner polynomials has the form

$$\begin{aligned} (T_{(g_l, g_r)} D_{m_1 m_2}^j)(u) &= D_{m_1 m_2}^j(g_l^{-1} u g_r) \\ &= \sum_{m'_1 m'_2} D_{m'_1 m'_2}^j(u) D_{m_1 m'_1}^j(g_l^{-1}) D_{m'_2 m_2}^j(g_r), \end{aligned} \quad (9)$$

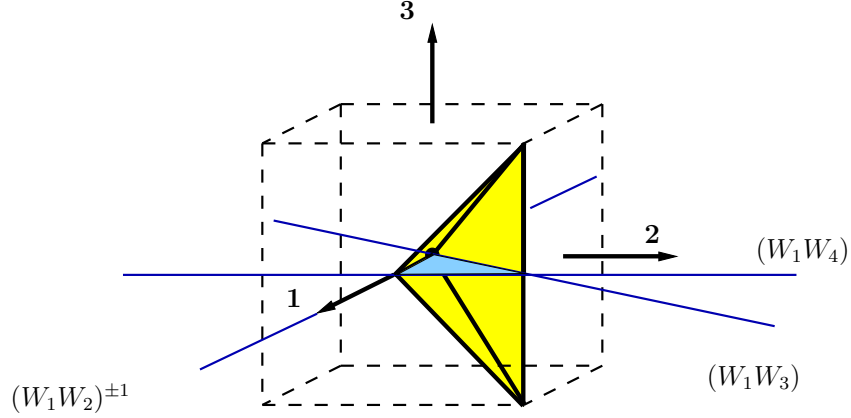


Figure 7. Cubic orbifold. Face gluing deck rotations of the cubic duplex orbifold (yellow) $N9$ in Euclidean version. Its four covering rotations (W_1W_3) , $(W_1W_2)^{\pm 1}$, (W_1W_4) , products of pairs of Weyl reflections from the Coxeter group with diagram $\Gamma = \circ - \overset{4}{\circ} - \circ - \circ$, are marked by rotation axes, intersection lines of pairs of Weyl planes. The three intersections of these axes with the orbifold form the edges of the glue triangle (light blue area). Its inner edge points have the orders $(2, 4, 2)$ of their covering rotations.

and so is determined by pairs of representation matrices of the group $SU(2, C)$. By eq. 9 we can take advantage of the elaborate techniques of angular momentum from [3].

Each spherical manifold carries a specific harmonic basis of eigenmodes, invariant under a deck group H , and obeying homotopic boundary conditions.

The eigenmodes of a spherical manifold offer two alternative views on different domains: (i) On a single closed manifold \mathcal{M} , they allow to expand square integrable observables. These obey homotopic boundary conditions for their functional values on faces and edges.

(ii) The covering 3-sphere is tiled by copies of \mathcal{M} . Any eigenmode now must on each different tile repeat its value, and fulfill the homotopic boundary conditions. It follows that the eigenmodes must display selection rules in comparison to a general polynomial basis on the 3-sphere. This second view allows for a comparison of observables for different topologies on the same domain, Einstein's 3-sphere.

In topology, these two views present (i) the local concept of homotopy of \mathcal{M} , and (ii) the concept of deck transformations from the group $H = \text{deck}(\mathcal{M})$ that generates the tiling on the 3-sphere as universal cover. Seifert and Threlfall [22] pp. 195-8 prove the equivalence of the two views: the fundamental or first homotopy group $\pi_1(\mathcal{M})$ that generates all the gluings is isomorphic to the group of deck transformations $\text{deck}(\mathcal{M})$ that generates the tiling on the cover.

4. Spherical Platonic 3-manifolds.

We study here the family of Platonic spherical 3-manifolds. For each Platonic polyhedron we construct in [12] on the universal cover, the 3-sphere, a unique group H of fixpoint-free deck transformations acting on the 3-sphere as a subgroup of a Coxeter group Γ . By the theorem from [22], the deck groups H are isomorphic to, and were derived in [12] from, the fundamental or first homotopy groups constructed by Everitt [5].

Coxeter groups Γ [2], [6] will become a main tool of the following analysis. They are generated on Euclidean 4-space with coordinates $x = (x_0, x_1, x_2, x_3)$, and on the 3-sphere, by four involutive Weyl reflections (W_1, W_2, W_3, W_4) in hyperplanes, see Table 2. A reflection hyperplane W_i containing the origin is characterized by a unit vector a_i perpendicular to it. The Coxeter

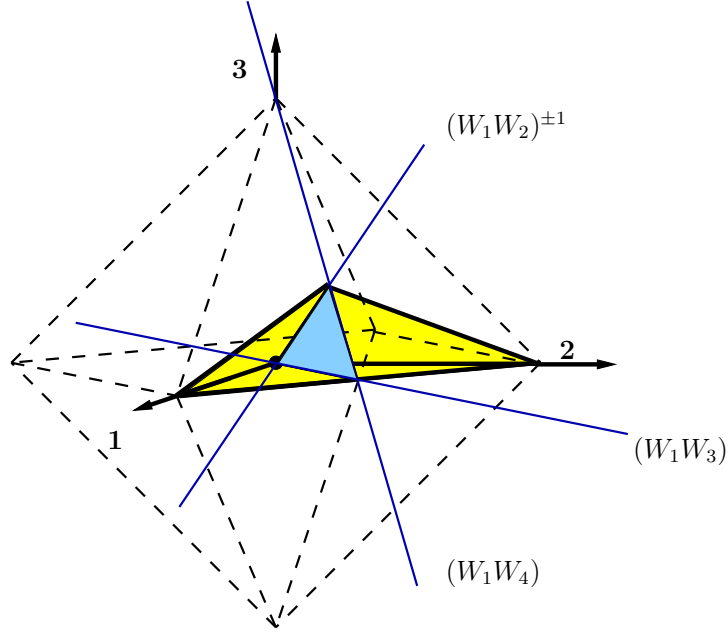


Figure 8. Octahedral orbifold. Face gluing deck rotations of the octahedral duplex orbifold (yellow) $N10$ in Euclidean version. The corresponding four products (W_1W_3) , $(W_1W_2)^{\pm 1}$, (W_1W_4) of pairs of Weyl reflections from $\Gamma = \circ - \circ \overset{4}{-} \circ - \circ$ are marked by rotation axes, intersection lines of pairs of Weyl reflection planes. The three intersections of these axes with the orbifold form the edges of the glue triangle (light blue area). Its inner edge points have the orders (2, 3, 2) of their covering rotations.

Coxeter diagram Γ	$ \Gamma $	polyhedron \mathcal{M}	$H = \text{deck}(\mathcal{M})$	$ H $	Reference
$\circ - \circ - \circ - \circ$	120	tetrahedron $N1$	C_5	5	[8]
$\circ - \overset{4}{\circ} - \circ - \circ - \circ$	384	cube $N2$ cube $N3$	C_8 Q	8 8	[9] [12]
$\circ - \circ - \overset{4}{\circ} - \circ - \circ$	1152	octahedron $N4$ octahedron $N5$ octahedron $N6$	$C_3 \times Q$ B \mathcal{T}^*	24 24 24	[12] [12] [12]
$\circ - \overset{5}{\circ} - \circ - \circ - \circ$	120^2	dodecahedron $N1'$	\mathcal{J}^*	120	[7]

Table 1. 4 Coxeter groups Γ , 4 Platonic polyhedra \mathcal{M} , 7 deck groups $H = \text{deck}(\mathcal{M})$ of order $|H|$. C_n denotes a cyclic, Q the quaternion, \mathcal{T}^* the binary tetrahedral, \mathcal{J}^* the binary icosahedral group. The symbols Ni are taken or extended from [5]. The last column gives the reference for the harmonic analysis. The diagram for the icosahedral Coxeter group is corrected compared to [12].

diagram encodes the group relations between the Weyl reflections, associated to its four nodes. A line connecting two nodes implies $(W_iW_{i+1})^3 = e$, a connecting line with integer superscript $k > 3$ implies $(W_iW_{i+1})^k = e$, Weyl reflections for nodes not connected by lines commute with one another. Tables 1, 2 review the relation of the Platonic polyhedra to Coxeter groups. In Table 2 we list the four unit vectors $a_j \in E^4$, perpendicular to the Weyl reflection hyperplanes,

Γ	a_1	a_2	a_3	a_4
$\circ - \circ - \circ - \circ$	$(0, 0, 0, 1)$	$(0, 0, \sqrt{\frac{3}{4}}, \frac{1}{2})$	$(0, \sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}, 0)$	$(\sqrt{\frac{5}{8}}, \sqrt{\frac{3}{8}}, 0, 0)$
$\overset{4}{\circ} - \circ - \circ - \circ$	$(0, 0, 0, 1)$	$(0, 0, -\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}})$	$(0, \sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}, 0)$	$(-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}, 0, 0)$
$\circ - \circ - \overset{4}{\circ} - \circ$	$(0, \sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}, 0)$	$(0, 0, -\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}})$	$(0, 0, 0, 1)$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
$\overset{5}{\circ} - \circ - \circ - \circ$	$(0, 0, 1, 0)$	$(0, -\frac{\sqrt{-\tau+3}}{2}, \frac{\tau}{2}, 0)$	$(0, -\sqrt{\frac{\tau+2}{5}}, 0, -\sqrt{\frac{-\tau+3}{5}})$	$(\frac{\sqrt{2-\tau}}{2}, 0, 0, -\frac{\sqrt{\tau+2}}{2})$

Table 2. The Weyl vectors a_s for the four Coxeter groups Γ from Table 1 with $\tau := \frac{1+\sqrt{5}}{2}$.

Coxeter diagram Γ	$ H $	polyhedron and orbifold	point group $M, M $	deck generators	fixpoint order k
$\circ - \circ - \circ - \circ$	5	tetrah N8	$A(4), 12$	$(W_1 W_3), (W_1 W_2)^{\pm 1}, (W_1 W_4)$	(2, 3, 2)
$\overset{4}{\circ} - \circ - \circ - \circ$	8	cube N9	$O, 24$	$(W_1 W_3), (W_1 W_2)^{\pm 1}, (W_1 W_4)$	(2, 4, 2)
$\circ - \circ - \overset{4}{\circ} - \circ$	24	octah N10	$O, 24$	$(W_1 W_3), (W_1 W_2)^{\pm 1}, (W_1 W_4)$	(2, 3, 2)
$\overset{5}{\circ} - \circ - \circ - \circ$	120	dodec N11	$\mathcal{I}, 60$	$(W_1 W_3), (W_1 W_2)^{\pm 1}, (W_1 W_4)$	(2, 5, 2)

Table 3. From four Coxeter groups Γ , $|S\Gamma| = |M| \cdot |H|$ and Platonic polyhedra to orbifolds. Point groups M , deck generators of orbifolds, and selected orders k for fixpoints on edges of the glue triangle of four duplex orbifolds N8, N9, N10, N11, see Figs. 6 -9. $A(4)$ is the tetrahedral, O the cubic, \mathcal{I} the icosahedral rotation group. The center of the polyhedron has order $|M|$, inner points of the orbifolds are regular. Note that the Weyl reflections W_i depend on the Coxeter group chosen, see Tables 1, 2.

for each Coxeter group. The Coxeter group tiles the 3-sphere into $|\Gamma|$ Coxeter simplices. The initial Platonic polyhedron consists of those Coxeter simplices which share a vertex at the center $x = (1, 0, 0, 0)$. In topology we prefer orientable manifolds. In the Coxeter groups this means that we must restrict our attention to the subgroups generated by an even number of Weyl reflections. We call the corresponding subgroups $S\Gamma$, where S stands for unimodularity of the defining representation on the Euclidean space E^4 . The order of these subgroups is $|S\Gamma| = |\Gamma|/2$. A set of generators of all unimodular Coxeter groups with four Weyl reflections is given by

$$S\Gamma : \{(W_1 W_2), (W_2 W_3), (W_3 W_4)\}. \quad (10)$$

The first two products of reflection generators generate the point group M of the Platonic polyhedron, the last product enters the group of deck operations of the Platonic tiling of the 3-sphere. Note that any product $(W_i W_j)$ leaves invariant the intersection of the two Weyl reflection hyperplanes perpendicular to the vectors $\{a_i, a_j\}$. On the 3-sphere this intersection is the geodesic rotation axis for the rotation $(W_i W_j)$. The representation of any such product is derived from their Weyl reflection vectors in [8] eq. (60): For two Weyl reflection operators with Weyl unit vectors $\{a_i, a_j\}$, define with eq. 6

$$v_i := u(a_i), \quad v_j := u(a_j). \quad (11)$$

Then the rotation operator for the product $(W_i W_j)$ is given by inserting $(g_l, g_r) = (v_i v_j^{-1}, v_i^{-1} v_j)$ into eq. 9,

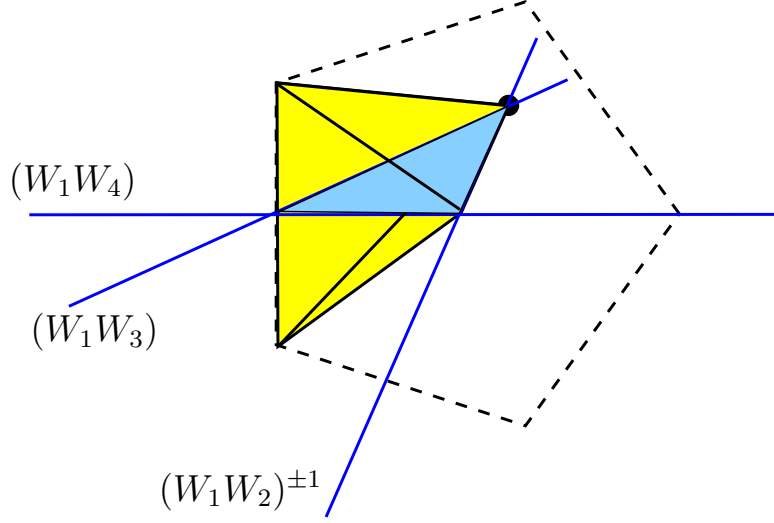


Figure 9. Dodecahedral orbifold. Face gluing deck rotations of the dodecahedral duplex orbifold (yellow) $N11$, attached to a dodecahedral face and the center, in Euclidean version. Only one face and the center are shown. The corresponding four covering products $(W_1W_3), (W_1W_2)^{\pm 1}, (W_1W_4)$ of two Weyl reflections from $\Gamma = \circ - \overset{5}{\circ} - \circ - \circ$ are marked by rotation axes, intersection lines of pairs of Weyl planes. The three intersections of these axes with the orbifold form the edges of the glue triangle (light blue area). Its inner edge points have the orders $(2, 5, 2)$ of their covering rotations.

$$T_{(W_iW_j)} = T_{(v_i v_j^{-1}, v_i^{-1} v_j)}. \quad (12)$$

As an example of an eigenmode we give from [12] the selection rules for the Platonic spherical cube manifold $N3$. The 3-sphere S^3 is tiled into eight spherical cubes. This tiling is the 8-cell. The deck transformations $H = Q$ from [12] form the quaternion group Q generated by the quaternions $\langle \pm e, \pm i, \pm j, \pm k \rangle$. The projector P_Q^0 to the identity representation of $H = Q$ applied to a Wigner polynomial is found to be

$$(P_Q^0 D_{m_1 m_2}^j)(u) = \frac{1}{8} [1 + (-1)^{2j}] [1 + (-1)^{m_1}] \times [D_{m_1 m_2}^j(u) + (-1)^{m_1} D_{-m_1 m_2}(u)], \quad (13)$$

it selects $j = \text{integer}, m_1 = \text{even}$.

5. Duplicates under the point group tile a polyhedron.

In [12] we introduce in addition for random functions the notion of random symmetry under the point group M of these manifolds. We argue there that the values of a proper random function on a polyhedral manifold whose shape displays the point symmetry M should be independent of operations from M . Therefore one should explore M for the analysis of the CMB radiation. The point groups are the tetrahedral group $A(4)$ for the tetrahedral 3-manifold, the cubic group O for the cubic and octahedral 3-manifolds, and the icosahedral group \mathcal{I} for the spherical dodecahedron. These point groups are unimodular subgroups w.r.t the 2-sphere

$$x_1^2 + x_2^2 + x_3^2 = 1, \quad (14)$$

and may be characterized on 3-space by unimodularity S and by Coxeter subdiagrams as

$$A(4) = S(\circ - \circ - \circ), O = S(\circ \overset{4}{-} \circ - \circ) \sim S(\circ - \circ \overset{4}{-} \circ), \mathcal{J} = S(\circ \overset{5}{-} \circ - \circ). \quad (15)$$

The action of the point group M on the Platonic polyhedron with center at $x = (1, 0, 0, 0)$ can now be decomposed into a pre-image and its images under M . The shape of a fundamental domain is not unique, but we can choose it in compact, convex and polyhedral form: Each fundamental domain for M we take as a duplex inside the proto-polyhedron, formed by gluing a Coxeter simplex and its mirror image under reflection in one simplex face. This simplex face we call the glue triangle. It is shown in Figs. 6, 7, 8, 9.

Prop 1: Fundamental domains under point groups: For each Platonic polyhedron, a duplex fundamental domain of their point group M may be chosen as shown in Figures 6, 7, 8, 9. All of them are again spherical simplices with four faces, have one vertex at the center of the polyhedron, and tile the polyhedron.

6. Orbifolds under the group ST tile the 3-sphere.

The Platonic 3-manifolds under their deck group H in turn tile the 3-sphere into $|H|$ copies of the Platonic polyhedron as prototile. These tilings into Platonic polyhedra are the m -cells, $m = |H|$, described in [23] and [12]. By composing the polyhedra, tiled into duplices, as tiles of the $|H|$ -cell tilings, it follows that the 3-sphere is tiled into $|ST| = |M| \cdot |H|$ duplices. With respect to actions on the 3-sphere, the Coxeter duplices, which originated as fundamental domains of the point group acting on the Platonic prototile, now become the fundamental domains of the much bigger groups ST . In the mathematical community they are called orbifolds. In Euclidean crystallography with space groups they are familiar under the name asymmetric units. A planar Euclidean example is provided in Fig. 5.

6.1. The action of deck and point groups.

The subgroup $H < ST$ of deck transformations for a given 3-manifold acts on the 3-sphere without fixpoints. The point group $M < ST$ by definition preserves the center of the polyhedron. It follows [14] that the intersection of these two subgroups contains only the identity, $H \cap M = e$. Any image of the orbifold as proto-duplex and pre-image under ST on the 3-sphere has a unique factorized address, composed of a unique point group element $p \in M$ acting on the initial polyhedron, followed by a unique deck transformation $h \in H$ from the initial polyhedron to an image on the $|H|$ -cell tiling. This leads to the following general conclusion on the group structure:

Prop 2: Unimodular Coxeter groups are products of subgroups : Given a prototile duplex, its image tile in the Coxeter duplex tiling of the 3-sphere results from the action of a group element from ST , uniquely factorized into a deck transformation from $h \in ST$ and a point transformation $p \in M$. This means that the elements $g \in ST$ obey the unique subgroup product law

$$g \in ST : g = hp, h \in H, p \in M, H \cap M = e, |ST| = |H| \cdot |M|. \quad (16)$$

Proof: The group ST with subgroups H, M all have faithful (one to one) representations, [25] p. 138, on E^4 . This implies that any image under $g \in ST$ of the orbifold as proto-duplex has a unique factorized address as a product $g = hp$, $h \in H, p \in M$. Similarly one can construct for the same image a unique factorized address $g = p'h'$.

This product structure $ST = H \cdot M$ differs from a direct or semidirect product. The two subgroups H, M do not commute, and the group H in general is not invariant under conjugation with elements from M , except in case of the cubic Coxeter group, analyzed in [12]. The

icosahedral rotation group for example has no invariant subgroup and so cannot be a semidirect product of two subgroups.

Eq. 16 shows that each of the subgroups generates the cosets for its partner.

6.2. Topology and deck groups for orbifolds.

We turn to the topological significance of the fundamental duplex domains for the group ST . A duplex of a unimodular Coxeter group on the 3-sphere cannot form a topological 3-manifold, because under the point group action it exhibits fixpoints of finite order $k > 1$ on its boundaries. The fixpoints are located on the rotation axis passing through the faces of the duplices, and their order is simply the order of the corresponding rotations.

The order k of a point is defined as the order of its stabilizer. Points of order $k = 1$ in topology are called regular, for order $k > 1$ singular, [21] pp. 664-6. We propose here to move in topology from the standard space forms to orbifolds. This concept is illustrated on the square in Fig. 5. We refer to Montesinos [20] pp. 78-97 and to Ratcliffe [21] pp. 652-714 for the introduction and mathematical terms associated with orbifolds. The notion includes a manifold structure and a covering, but admits singular points of finite order.

We claim:

Prop 3: Spherical Platonic 3-manifolds with point symmetry are appropriately described as orbifolds.

We shall demonstrate orbifold coverings by deck transformations from ST , and derive the harmonic analysis for their use in cosmic topology.

In figures 6, 7, 8, 9 we reproduce, with minor changes, based on [12], [13], a set of fundamental duplex domains for the point groups. Now we interpret them as 3-orbifolds under ST acting on the 3-sphere, based on the tetrahedron, the cube, the octahedron or the dodecahedron, drawn in their Euclidean version. The edges of the Platonic polyhedra are given in dashed lines, the orbifolds are marked by yellow color. Note that the polyhedra in the figures are drawn in Euclidean fashion, but stand for spherical polyhedra on S^3 .

For topological 3-manifolds, the deck group on the 3-sphere is generated by the operations which map the pre-image of the manifold to all its face neighbours. We present a corresponding analysis for the deck group of an orbifold on the 3-sphere.

We use the tetrahedral manifold $N2$ from Table 3 for the demonstration of orbifold covering. Here the Coxeter group is $S(5)$, the symmetric group on 5 objects, with unimodular subgroup $ST = A(5)$, the group of even permutations. In Fig. 6 we mark the four vertices of the tetrahedron by the numbers 1, 2, 3, 4. A different enumeration is used in [12], Fig. 4. The Weyl reflection operators are in one-to-one correspondence to the vertex transpositions

$$W_1 = (1, 2), W_2 = (2, 3), W_3 = (3, 4), W_4 = (4, 5). \quad (17)$$

From eq. 17 and Fig. 6 we find for the four covering deck rotations of this orbifold the simple expressions

$$\begin{aligned} (W_1 W_3) &= (1, 2)(3, 4), \\ (W_1 W_2) &= (1, 2)(2, 3) = (1, 2, 3), (W_1 W_2)^{-1} = (2, 3)(1, 2) = (3, 2, 1), \\ (W_1 W_4) &= (1, 2)(4, 5), \end{aligned} \quad (18)$$

compare Fig. 6. The three even permutations in eq. 18 cover the orbifold and generate $A(5)$. We return to all four orbifolds associated with Platonic polyhedra and denote them by $N8, N9, N10$ as in [13] and by $N11$ for the Platonic dodecahedron. Each face of the orbifold as proto-tile of the duplex tiling is covered by a face-neighbour, a copy of the orbifold. Instead of drawing the neighbours, we give in the Figures 6-9 the deck rotations from ST which map the preimage into

its face neighbours. These deck rotations have axes which intersect with faces of the orbifold. In the figures we denote the rotation axes by blue lines and give in each case the rotation as an even product of Weyl reflections. The first three deck rotations all have axes passing through the center of the polyhedron and generate the point group M . The last even product always involves a Weyl reflection that passes through an outer face of the Platonic polyhedron. The rotation containing this Weyl reflection has order $k = 2$. It transforms the orbifold into a face neighbour inside a new Platonic neighbour polyhedron. By examining the deck operations from the figures and comparing them with the Coxeter group relations, it can be verified in each case that the four deck operations that cover the faces of the orbifold generate the full unimodular Coxeter group ST . This is in full analogy to the role of deck groups H for spherical 3-manifolds [12].

The intersections of the covering rotation axes with the 3-orbifold determine singular points and their orders. The center point of the initial spherical polyhedron, chosen as $x = (1, 0, 0, 0)$ and marked in the figures in black, is a fixpoint of maximal order $k = |M|$. For the orbifolds under inspection, more fixpoints appear on the inner points of edges of the glue triangle of each duplex, with area marked in the figures in light blue. The order k of these fixpoints agrees with the order of the covering rotation. Note that, in contrast to the Euclidean drawings, we always infer the order k of the rotations from the Coxeter group relations and their spherical settings on the 3-sphere. We give the covering rotations and orders of these fixpoints in Table 3.

The obvious rotation axes from the point group M determine additional singular points on inner positions of edges of the orbifolds. For all four orbifolds we find by inspection:

Prop 4: Deck transformations of orbifolds generate unimodular Coxeter groups:

Deck rotations cover the four faces of the four Platonic 3-orbifolds, they generate the duplex tiling of S^3 by all elements of the corresponding unimodular Coxeter deck groups ST . It follows that the orbifolds can abstractly be characterized as quotient spaces S^3/ST .

7. Harmonic analysis on orbifolds.

The harmonic analysis can follow the two views described in section 3. Clearly the homotopic boundary conditions imply selection rules compared to the full basis on the 3-sphere. The CMB radiation on an assumed topology usually is modelled by random coefficients in the specific basis. Here we analyze the basis construction, but postpone any numerical modelling.

On the 3-sphere, after replacing the Cartesian coordinates x by matrix coordinates $u = u(x) \in SU(2, C)$ [12], the harmonic basis may be spanned by Wigner polynomials $D_{m_1 m_2}^j(u)$ [3]. More convenient for pure three-dimensional rotations of 3-space (x_1, x_2, x_3) as they appear in the point groups are the spherical polynomials $\psi(jlm)(u)$, linearly related to the Wigner polynomials by

$$\begin{aligned} \psi(jlm)(u) &= \sum_{m_1 m_2} D_{m_1 m_2}^j(u) \langle j - m_1 j m_2 | lm \rangle (-1)^{(j-m_1)}, \\ D_{m_1 m_2}^j(u) &= \sum_{lm} \psi(jlm)(u) \langle j - m_1 j m_2 | lm \rangle (-1)^{(j-m_1)}, \quad 0 \leq l \leq 2j, \end{aligned} \quad (19)$$

and with summations restricted by the Wigner coefficients [3]. The spherical polynomials transform, under rotations of the last three coordinates (x_1, x_2, x_3) while preserving x_0 , like the spherical harmonics Y_m^l [12],

$$(T_{(g_l, g_l)} \psi)(jlm)(u) = \psi(jlm)(g_l^{-1} u g_l) = \sum_{m'=-l}^{m'=l} \psi(jlm')(u) D_{m', m}^l(g_l). \quad (20)$$

They are adapted to the application of rotations from the point group. Point symmetry selects a lowest multipole order l_0 , compare [12] Table 3, [13]. The Wigner polynomial basis is suited

for the projection of the subbasis invariant under the deck group H . In the cubic case, the deck group of the 3-manifold $N3$ is the quaternion group $H = Q$. Invariant under conjugation with the cubic point group O , it forms a semidirect product $ST = (Q \times_s O)$.

The harmonic analysis on the four spherical orbifolds $N8, N9, N10, N11$ is given by polynomials invariant under their unimodular Coxeter group ST . These and only these polynomials repeat their values on any Coxeter duplex from the tiling.

Prop 5: Harmonic analysis on spherical 3-orbifolds: The harmonic analysis on a spherical orbifold is spanned by harmonic polynomials, invariant under its group of deck transformations ST .

We have seen in Prop. 2 that any element of ST admits a unique factorization. For the projectors to the identity representation of ST we claim by use of eq. 16:

Prop 6: Factorization of projectors: The projector to the identity representation, denoted by Γ_1 , for the unimodular Coxeter group ST factorizes into the product of the projectors to the identity representations Γ_1 for the two subgroups H, M ,

$$P_{ST}^{\Gamma_1} = P_H^{\Gamma_1} P_M^{\Gamma_1} = P_M^{\Gamma_1} P_H^{\Gamma_1}. \quad (21)$$

Proof: In the group operator algebra of the unimodular Coxeter group ST we have from eq. 16 the factorization

$$\begin{aligned} P_{ST}^{\Gamma_1} &= \sum_g T_g = \sum_{h,p} T_{hp} = \sum_{p,h} T_{ph}, \quad h \in H, p \in M, \\ &= \sum_{h,p} T_h T_p = \left(\sum_h T_h \right) \left(\sum_p T_p \right) = \left(\sum_p T_p \right) \left(\sum_h T_h \right) = P_H^{\Gamma_1} P_M^{\Gamma_1} = P_M^{\Gamma_1} P_H^{\Gamma_1}. \end{aligned} \quad (22)$$

This result greatly simplifies the projection to the identity representation:

The projectors $P_H^{\Gamma_0}$ for the group H of deck transformations are given in [12] in the Wigner polynomial basis, see eq. 19.

Prop 7: Orbifolds give sharp multipole selection rules: ST -invariant polynomials by their point group M -invariance select a lowest non-zero multipole order l_0 , see [12] Table 3. This projection is carried out in [13] for the spherical cubic 3-orbifold and multipole order $0 \leq l \leq 8$. The results are reproduced, and in part corrected compared to [13], in Tables 4 and 5 in explicit multipole order.

We note that the results depend on a convenient choice of the coordinates in the spherical harmonics with respect to the symmetry axes of the orbifolds. Under a general rotation of the system of these coordinates one would obtain many additional terms of the same multipole order l . The function $R_{2j+1,l}(\chi)$ is a Gegenbauer polynomial given in [12] eq. (55). The analysis can be extended by the method described in section 9.

A similar analysis applies to the Platonic spherical tetrahedral, octahedral and dodecahedral 3-orbifolds.

8. Homotopic boundary conditions from orbifolds.

From [12] we know that a fixed geometric shape of a Platonic 3-manifold can have different and inequivalent topologies, characterized by different groups H of homotopies and deck transformations. These differences give rise to different homotopic boundary conditions. We now examine the boundary conditions for orbifolds.

We have seen that the orbifold is covered face-to-face by its rotational images. It follows, as in the case of 3-manifolds, that the topology on 3-manifolds implies homotopic boundary conditions on the faces of the 3-orbifolds. Since with the orbifolds we introduce point symmetry in addition to deck transformations, we find from the arguments given in [12] for random point symmetry,

l	$Y^{\Gamma_1, l} = \sum_m a_{lm} Y_m^l(\theta, \phi)$
0	Y_0^0
4	$\frac{1}{2}\sqrt{\frac{7}{3}}Y_0^4 + \frac{1}{2}\sqrt{\frac{5}{6}}(Y_4^4 + Y_{-4}^4)$
6	$\frac{1}{2\sqrt{2}}Y_0^6 - \frac{\sqrt{7}}{4}(Y_4^6 + Y_{-4}^6)$
8	$\frac{\sqrt{33}}{8}Y_0^8 + \frac{1}{4}\sqrt{\frac{7}{6}}(Y_4^8 + Y_{-4}^8) + \frac{1}{8}\sqrt{\frac{65}{6}}(Y_8^8 + Y_{-8}^8)$

Table 4. The first cubic invariant spherical harmonics $Y^{\Gamma_1, l}$, expressed by spherical harmonics $Y_m^l(\theta, \phi)$. Γ_1 denotes the identity representation of the point group H .

$2j$	l	$\psi^{0, \Gamma_1, 2j} = \sum_l b_l R_{2j+1, l}(\chi) Y^{\Gamma_1, l}(\theta, \phi)$
0	0	$R_{10} Y^{\Gamma_1, 0}$
4	0, 4	$\sqrt{\frac{2}{5}} R_{50} Y^{\Gamma_1, 0} + \sqrt{\frac{3}{5}} R_{54} Y^{\Gamma_1, 4}$
6	0, 4, 6	$\sqrt{\frac{1}{7}} R_{70} Y^{\Gamma_1, 0} + \sqrt{\frac{6}{11}} R_{74} Y^{\Gamma_1, 4} + 2\sqrt{\frac{6}{77}} R_{76} Y^{\Gamma_1, 6}$
8	0, 4, 6, 8	$\sqrt{\frac{1}{57}} R_{90} Y^{\Gamma_1, 0} + 9\sqrt{\frac{2}{2717}} R_{94} Y^{\Gamma_1, 4}$ $+ 2\sqrt{\frac{38}{165}} R_{96} Y^{\Gamma_1, 6} + \sqrt{\frac{2}{1235}} R_{98} Y^{\Gamma_1, 8}$

Table 5. Harmonic analysis on the cubic Platonic orbifold $N3$: The first $(S\Gamma = (Q \times_s O))$ -invariant polynomials $\psi^{0, \Gamma_1, 2j}$ of degree $2j = 0, 4, 6, 8$ on the 3-sphere, expressed by Gegenbauer polynomials $R_{nl}(\chi)$ and cubic invariant spherical harmonics $Y^{\Gamma_1, l}$ from Table 4. $(Q \times_s O)$ -invariance enforces superpositions of several cubic invariant spherical harmonics. Note the suppression of odd and $l = 2$ multipole orders in the eigenmodes.

Prop 8: Topological universality from point symmetry: If we demand, for a function on a given Platonic 3-manifold, point symmetry under M in addition to the boundary conditions set by homotopy on faces and edges, then new boundary conditions apply universally, that is, dependent on the point group M but independent of the specific group H of deck transformations chosen.

Prop 9: Universal homotopic boundary conditions from 3-orbifolds: If in addition to homotopy we demand on the manifold symmetry under the rotational point group M , the homotopic boundary conditions for different deck and homotopy groups on the same Platonic geometrical shape coincide with one another and reduce to the homotopic boundary conditions for the orbifolds. Their boundary conditions are determined by the generators in Table 1 of the covering rotations.

9. Recursive computation of the bases invariant under the orbifold deck groups $S\Gamma = H \cdot M$.

We describe the recursive construction of $S\Gamma$ -invariant bases by use of the factorization given in Prop. 4. This recursive construction is used to obtain the results of Tables 4, 5. The multiplicity $m(l, \Gamma_1)$ of the identity representation Γ_1 of M for given multipole order l is given in [17] pp. 436-8.

Our basic tool are the relations [12] between Wigner and spherical harmonic polynomials on the 3-sphere given in eq. 19.

(i) We start from a linear combination of spherical harmonics of multipole order l , invariant under the point group M , and construct with a Gegenbauer polynomial $R_{nl}(\chi)$ and its coefficients a_{lm} a M -invariant linear combination of spherical polynomials $\psi_{jlm}(u)$, $l \leq 2j$ eq. 22, on the 3-sphere

[12] with lowest degree $l = l_0 = 2j$,

$$\psi_{j,l}^0(u) = \sum_m^l a_{lm'} \psi_{j,l,m'}(u). \quad (23)$$

By the upper index 0 we denote invariance under the point group M . The starting linear combinations of spherical harmonics for lowest multipole order $l_0 > 0$ for the point groups involved can be found in the literature. For tetrahedral point symmetry the lowest multipole order from [12] Table 3 is $l_0 = 3$. For cubic point symmetry $M = O$ this function has $l_0 = 4$ and is given in Table 4. Since the point group M acts by rotations eq. 20 only, the coefficients $a_{lm'}$ in eq. 23 are independent of the degree $n = 2j$ of the polynomial.

(ii) Next we transform with eq. 19 from the spherical to the Wigner basis and apply the projector $P_H^{\Gamma_1}$ for H -invariance, for example the one with $H = Q$ for the cubic manifold $N3$ given in eq. 13. If the resulting function does not vanish, we transform with eq. 19 back to the spherical basis. In this way we find from the starting function a new one, invariant from Prop. 4 under both M and H and hence under ST , given by

$$\begin{aligned} \psi_{j,\Gamma_1}(u) &= \sum_i \left[\sum_m a_{l+i,m}^{\Gamma_1} \psi_{j,l+i,m}(u) \right], \\ a_{l+i,m}^{\Gamma_1} &= \sum_{m',m'_1,m'_2,m_1,m_2} a_{lm'} \langle j - m'_1 j m'_2 | l m' \rangle (-1)^{(j-m'_1)} \\ &\quad \langle j m'_1 m'_2 | P_H^{\Gamma_1} | j m_1 m_2 \rangle \langle j - m_1 j m_2 | l + i m \rangle (-1)^{(j-m_1)}. \end{aligned} \quad (24)$$

The polynomial eq. 24, if non-vanishing, can be normalized.

The Wigner coefficients are given in [3], and the matrix elements of the projector $P_H^{\Gamma_1}$ for the identity representation of the group H are given in the Wigner polynomial basis by

$$\langle j m'_1 m'_2 | P_H^{\Gamma_1} | j m_1 m_2 \rangle = \frac{1}{|H|} \sum_{h=(h_l, h_r) \in H} D_{m'_1 m_1}^j(h_l^{-1}) D_{m'_2 m_2}^j(h_r), \quad (25)$$

and specified in [12] for each Platonic 3-manifold. Due to universality Prop. 5, we can choose the most convenient deck group H for a fixed geometric shape. The recursion relation eq. 24 involves Wigner coefficients, the elements of the group H given as pairs $h = (h_l, h_r)$ in [12], and Wigner D^j -representations for the group $SU(2, C)$. The ST -invariant bases appear as linear combinations of M -invariant spherical functions with fixed multipole order $l + i$.

(iii) Moreover, since the point group action cannot change the multipole order, each new partial sum of eq. 24 in square brackets for fixed $l + i$ must generate a (new) invariant under the point group M . This allows to restart the computation with $l \rightarrow l + i$ by going again from spherical to Wigner polynomials, followed by projection of an invariant under H . In this way we can increase the polynomial degree $2j$. In Tables 4 and 5 we show the first steps in the recursive computation of eigenmodes for the group $Q \times_s O$. By character technique, see [8], [9], we can control the number of invariants for given degree $2j$ of the polynomials.

10. Applications.

Before applying the eigenmodes as found in Tables 4, 5 to the analysis of CMB radiation, one has to consider several questions from astrophysics. In principle the CMB radiation spectrum reflects the density fluctuations of the primordial plasma at the time of last scattering. In the cosmological process of creation and propagation of the CMB radiation one has to consider the Sachs-Wolfe and the Doppler effect. An obstacle is a strong foreground contamination of the observed radiation due to CMB sources within our own galaxy, see [24], which the WMAP team tried to eliminate. The Ulm group of physicists has experience in incorporating all these astrophysical data. The standard procedure then is to choose a topological manifold, to consider random superpositions of its eigenmodes up to a maximal polynomial degree and multipole order, to incorporate the astrophysical data, and compute spectra similar to Fig. 1. An application of the cubic Platonic 3-manifold $N3$ that includes all astrophysical data is given in cooperation with the Ulm group in [1].

11. Conclusion.

In the present analysis, we follow J Levin in [18] p. 305, who in the context of hyperbolic 3-manifolds sees no convincing reason to omit the study of orbifolds in relation to the CMB. We illuminate the topological background of cosmic topology and then develop new methods of group representations for the analysis of eigenmodes for spherical orbifolds. We see the future interpretation and use of spherical 3-orbifolds from Table 1 in cosmic topology along two lines: (I) We can retain the strict original notion and analysis of topological 3-manifolds. The new harmonic basis characterized by point symmetry forms a subbasis of the harmonic analysis based on the deck group H . If this subbasis fails to model the CMB fluctuations, one can return to the larger basis of [12] for the 3-manifold without point symmetry.

(II) We formulate the notion of a topological 3-manifolds with (random) point symmetry M in terms of spherical 3-orbifolds. The relevant groups of deck transformations are shown to be four unimodular Coxeter groups ST . They generate, from a Coxeter duplex orbifold as prototile, a tiling of the 3-sphere into $|ST|$ copies. We factorize these groups as $ST = H \cdot M$ into products of a group H of polyhedral deck transformations with the point group M of the polyhedron. The Platonic 3-manifolds under the assumption of random point symmetry are replaced by 3-orbifolds, and their harmonic bases are supported on a fraction $1/|ST|$ of the 3-sphere.

For the deck group $ST = H \cdot M$ we construct a new harmonic analysis which can model the CMB radiation. Point symmetry implies topological universality: all fundamental groups for the same geometrical polyhedral shape produce the same boundary conditions. In contrast to general spherical space forms, point symmetry enforces, depending on the chosen point symmetry, a sharp non-zero onset of the multipole expansion of CMB radiation with harmonic polynomials of minimal non-trivial degree $2j_0 = l_0$. This phenomenon is in line with the physics of molecules with point symmetry [17]. The multipole selection rules, if present, must show up already in the lowest multipole orders of the CMB radiation. Moreover, topology correlates polynomials of increasing multipole order with one another. These phenomena are exemplified in Tables 4, 5 for orbifolds from spherical cubes.

Acknowledgement: The author is grateful to Tobias Kramer for the recursive algebraic computation of the results given in Tables 4 and 5.

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